EM my note

（可能有一些笔误，请参考教材）

$$\begin{cases} X^\mu \equiv (t, \vec{x}) \\ X_\mu = (t, -\vec{x}) \end{cases} \qquad \eta^{\mu\nu} = (+, -, -, -)$$

$$\boxed{X_\mu = \eta_{\mu\nu} X^\nu}$$

$$\begin{cases} A^\mu & \text{逆变} \\ A_\mu & \text{协变} \end{cases}$$

$$\begin{cases} A^\mu = (\phi, \vec{A}) & j^\mu = (\rho, \vec{J}) \\ A_\mu = (\phi, -\vec{A}). & j_\mu = (\rho, -\vec{J}) \end{cases}$$

<u>常见代数技巧</u>

$$\vec{a} \times \vec{b} = a_i b_j \varepsilon^{ijk} e_k$$
$$= \varepsilon^{ijk} a_i b_j e_k$$

$$(\nabla \times \vec{b}) \equiv \varepsilon^{ijk} (\partial_i b_j) e_k$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c}) \ ?$$

$$\vec{a} \times (\vec{b} \times \vec{c}) \equiv \vec{a} \times b_i C_j \varepsilon^{ijk} e_k$$
$$= (\vec{a} \times e_k) b_i C_j \varepsilon^{ijk}$$
$$= \varepsilon^{ijk} b_i C_j (\vec{a} \times e_k) \qquad \text{so } \beta = k$$
$$= \varepsilon^{ijk} b_i C_j a_\alpha (e_k)_\beta \varepsilon^{\alpha\beta\gamma} e_\gamma$$

$$= \underbrace{\varepsilon^{ijk} b_i c_j a_l \varepsilon}^{lks} e_s$$

$$= \underline{\varepsilon^{ijk} \varepsilon^{lks}} a_l b_i c_j e_s$$

两个 ε 的 ~~[illegible]~~ 乘积，如果见到2个 ε 的乘积，则要联想到 $(a \times b \times c)$ OR $(\nabla \times \nabla \times \vec{a})$.

$$\begin{cases} \partial^{\mu} = ? \\ \partial_{\mu} = ? \end{cases}$$

$$f(x) = f_0 + \frac{\partial f}{\partial x} x$$

so $f(x^i) \equiv f_0 + \frac{\partial f}{\partial x^i} x^i = f_0 + (\partial_i f) x^i$

$$\Rightarrow \boxed{\partial_i \equiv \frac{\partial}{\partial x^i}}$$

同样 $\partial^i \Rightarrow \frac{\partial}{\partial x_i}$ 因为

$$f(x_i) \equiv f_0 + \frac{\partial f}{\partial x_i} x^i \equiv \underline{f_0 + (\partial^i f) x_i}$$

以後用 Einstein summation.

$$F^{0i} = \eta^{0\mu}_{0} F_{\mu\nu} \eta^{\nu i}$$

so OR $F^{\mu\nu} = \eta^{\mu\mu'} \eta^{\nu\nu'} F_{\mu'\nu'}$ $\qquad \eta^{\alpha\beta} = \eta^{\beta\alpha}$

$$\Leftrightarrow \quad F^{01} = \eta^{00} F^{11} F_{01} = (+1)(-1) F_{01} = -F_{01}$$

OR $F^{0i} = -F_{0i}$ for $i = 0 - 3$.

$$F^{ij} = \eta^{i\alpha} \eta^{j\beta} F_{\alpha\beta}$$

$$= (-1)(-1) F_{\alpha\beta} = \boxed{F_{ij}}$$

$\alpha \to i$

So $\boxed{F^{ij} = F_{ij}}$

$$\vec{B} = \nabla \times \vec{A}$$

$$B^k = \frac{\partial}{\partial x^i} A^j \, \varepsilon^{ijk}$$

$$= \varepsilon^{ijk} \partial_i A^j$$

$$= -\varepsilon^{ijk} \partial_i A_j \quad \text{using } A^j = -A_j$$

$$= -\frac{1}{2} \varepsilon^{ijk} [\partial_i A_j - \partial_j A_i]$$

$$= -\frac{1}{2} \varepsilon^{ijk} F_{ij} .$$

$$B_k = \left(\frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i\right)$$

$$= \varepsilon_{ijk}\, \partial^i A_j$$

$$= -\varepsilon_{ijk}\, \partial^i A^j$$

$$= -\frac{1}{2} \varepsilon_{ijk} \left(\partial^i A^j - \partial^j A^i\right)$$

$$= -\frac{1}{2} \varepsilon_{ijk} F^{jk}$$

Summary

$$\begin{cases} E^i = F_{0i} = -F^{0i} = F^{i0} \\ E_i = -F^{0i} = \end{cases}$$

$$\begin{cases} B^k = -\frac{1}{2} \varepsilon^{ijk} F_{ij} \\ B_k = -\frac{1}{2} \varepsilon_{ijk} F^{ij} \end{cases}$$

$\nabla \cdot \vec{E} = \rho \Rightarrow \underline{\partial_i E^i} = \rho$

using $\underline{E^i = F_{0i} = -F^{0i} = F^{i0}}$

$$\boxed{\partial_i F^{i0} \equiv j^0}$$

$$\boxed{\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J}}$$

$$J^k \equiv (\nabla \times \vec{B})^k - \left(\frac{\partial}{\partial t}\vec{E}\right)^k$$

$$= \varepsilon^{ijk}\left(\frac{\partial}{\partial x^i} B^j\right) - \frac{\partial}{\partial t} E^k$$

$$= \varepsilon^{ijk} \partial_i B^j - \frac{\partial}{\partial x^0} E^k$$

$$= \varepsilon^{ijk}(\partial_i B_j) - \partial_0 E^k$$

$$= \varepsilon^{ijk}(\partial_i B_j) - \partial_0 E^k$$

$$= \varepsilon^{ijk} \partial_i (B_j) + (\partial_0 F^{0k})$$

$$B_j \equiv -\frac{1}{2}\varepsilon_{j\alpha\beta} F^{\alpha\beta}$$

$$= -\frac{1}{2}\underbrace{\varepsilon^{ijk} \varepsilon_{j\alpha\beta}} \partial_i F^{\alpha\beta} + \partial_0 F^{0k}$$

$$= \frac{1}{2}\varepsilon^{jik} \varepsilon_{j\alpha\beta} \underbrace{\partial_i F^{\alpha\beta}} = \frac{1}{2}\partial_i F^{ik} - \frac{1}{2}\partial_i F^{ki} + \partial_0 F^{0k}$$

α, β

$$\equiv \underbrace{\partial_i F^{ik}} = \partial_\mu F^{\mu k} \equiv j^k$$

結合上面2个方程

$$\begin{cases} \nabla \cdot \vec{E} = \rho \\ \nabla \times \vec{B} = \dfrac{\partial \vec{E}}{\partial t} + \vec{J} \end{cases}$$

Unified by

$$\begin{cases} \partial_\mu F^{\mu k} = j^k \\ j = (\rho, \vec{J}) \end{cases}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ [illegible]

$$\Rightarrow \quad (\nabla \times \vec{E})^k = -\frac{\partial B^k}{\partial t}$$

$$= -\frac{\partial}{\partial x^0} B^k = -\partial_0 B^k$$

$$= -\frac{1}{2} \varepsilon^{k\alpha\beta} (\partial_0 F_{\alpha\beta})$$

$$\varepsilon^{ijk} \partial_i E^j_0 \qquad E^j = F_{0j}$$

$$= -\varepsilon^{ijk} \partial_i E_j$$

$$\Leftrightarrow \quad \varepsilon^{ijk} \partial_i F_{0j} = \frac{1}{2} \varepsilon^{k\alpha\beta} \partial_0 F_{\alpha\beta}$$

$$\underbrace{\varepsilon^{ijkl} \partial_j F_{kl}} \equiv 0 \,!$$ [illegible]

① $i = 0$

② $i = x$

$$\varepsilon^{1jkl} \partial_j F_{kl} = 0$$

① $j =$

$$\underline{\varepsilon^{10kl} \partial_0 F_{kl}}$$

$$+ \underline{\varepsilon^{1j0l} \partial_j F_{0l}}$$

$$+ \underline{\varepsilon^{1jk0} \partial_j F_{k0}} \equiv 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\Rightarrow \quad \partial_i B^i = 0$$

$$B^i \equiv -\tfrac{i}{2}\, \varepsilon^{ijk} F_{jk}$$

$\vec{B}$: (1 - 3)

$$\Leftrightarrow \quad -\tfrac{i}{2}\, \varepsilon^{ijk}\, \partial_i F_{jk} \equiv 0$$

OR

$$\boxed{\varepsilon^{ijk}\, \partial_i F_{jk} \equiv 0}$$

$$F = F_{jk}\, dx^j \wedge dx^k$$

用 2-form 表示

$$dF \equiv \frac{\partial F_{jk}}{\partial x^l}\, dx^l \wedge dx^j \wedge dx^k$$

$$= \varepsilon^{0ljk}\, \partial_l F_{jk} \equiv \boxed{0}\ ?$$

$$\boxed{\varepsilon^{ijkl}\partial_j F_{kl} \equiv 0}$$

① $i=0$ 일때

$$\varepsilon^{0jkl}\partial_j F_{kl} \equiv 0$$

$$\Rightarrow \varepsilon^{0123}\partial_1 F_{23} + \varepsilon^{0132}\partial_1 F_{32}$$ [illegible]

$$+ \varepsilon^{0213}\partial_2 F_{13} + \varepsilon^{0231}\partial_2 F_{31}$$

$$+ \varepsilon^{0312}\partial_3 F_{12} + \varepsilon^{0321}\partial_3 F_{21} \equiv 0$$

$$\partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12} = 0.$$

using $B^j = -\frac{1}{2}\varepsilon^{ijk} F_{jk}$

$$\Leftrightarrow \begin{cases} B^1 = -F_{23} & 123 \\ B^2 = F_{13} & 213 \\ B^3 = -F_{12} & 312 \end{cases}$$

$$\Rightarrow -\partial_1 B^1 - \partial_2 B^2 - \partial_3 B^3 = 0 \Leftrightarrow \nabla\cdot\vec{B} = 0.$$

② $i=1$ 일때

$$\boxed{E^j = F_{0j}}$$

$$\underline{\varepsilon^{1023}\partial_0 F_{23}} + \underline{\varepsilon^{1032}\partial_0 F_{32}}$$

$$+ \varepsilon^{1230}\partial_2 F_{30} + \cdots$$

$$+ \varepsilon^{1320}\partial_3 F_{20} + \cdots = 0.$$

0123
1230
2301
3012

$$\Rightarrow \boxed{-\partial_0 F_{23} + \partial_2 F_{03} + \partial_3 F_{02} = 0} \quad \Bigg\} \quad \boxed{\nabla\times E = -\frac{\partial B}{\partial t}}$$

$$\Leftrightarrow +(\partial_t B^1) - \partial_y E^z + \partial_z E^y = 0.$$

$F_{\mu\nu} \rightarrow F^{\mu\nu}$

Duality

$\downarrow$

$\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2}\mathcal{E}^{\mu\nu\alpha\beta} F_{\alpha\beta}} \Rightarrow \tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu}$

反对称

eg $\tilde{F}^{01} \equiv \frac{1}{2}\mathcal{E}^{01\alpha\beta}F_{\alpha\beta}$

$= \frac{1}{2}\mathcal{E}^{0123}F_{23} + \frac{1}{2}\mathcal{E}^{0132}F_{32}$

$= \mathcal{E}^{0123}F_{23} = F_{23}$

$\tilde{F}^{01} \longleftrightarrow F_{23}$

E B

eg $\tilde{F}^{12} = \frac{1}{2}\mathcal{E}^{1203}F_{03} + \frac{1}{2}\mathcal{E}^{1230}F_{30}$

$= -F_{03}$

$\tilde{F}^{12} \longleftrightarrow F_{03}$

$\boxed{B} \longleftrightarrow \boxed{E}$

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use $\tilde{F}^{\mu\nu} = \frac{1}{2}\mathcal{E}^{\mu\nu\alpha\beta}F_{\alpha\beta}$

$\mathcal{E}^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta} \equiv \partial_\nu(\mathcal{E}^{\mu\nu\alpha\beta}F_{\alpha\beta})$

$= \partial_\nu \tilde{F}^{\mu\nu} \equiv 0. \Rightarrow \boxed{\partial_\nu \tilde{F}^{\nu\mu} \equiv 0}$

Lorentz force

11

$\leftarrow$ 2 is $\frac{1}{10}$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \rho\vec{E} + \vec{J} \times \vec{B}$$

$$\vec{F} = f\,dV \quad q = \rho\,dV \quad \rho\vec{v} = \vec{J}$$

$$(f)' = \rho E' + (\vec{J} \times \vec{B})' \qquad \rho \to j^0$$

$$= \rho E' + (J^2 B^3 - J^3 B^2)$$

$$= \underbrace{j^0 E^{01}} + J^2 B^3 - J^3 B^2$$

$$\boxed{B^i = -\frac{1}{2}\varepsilon^{ijk} F_{jk}}$$

$$= j^0 F^{01} - J^2 F_{12} - J^3 F_{13}$$

$$B^1 = -F_{23} \qquad 123$$

$$B^2 = +F_{13} \qquad 231$$

$$B^3 = -F_{12} \qquad 312$$

$F^{01} = -F_{01}$

$$= -j^0 F_{01} - J^2 F_{12} - J^3 F_{13}$$

$$= -j^0 F_{01} - j^1 F_{11} - j^2 F_{12} - j^3 F_{13}$$

$$\boxed{f^1 = -j^\mu F_{\mu 1}}$$

force can also be expressed in terms of $F_{\mu\nu}$ tensor.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

why $\frac{1}{4}$?

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & & & \\ -E_y & & \vec{B} \cdots & \\ -E_z & & & \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & & & \\ E_y & & \vec{B} & \\ E_z & & & \end{pmatrix}$$

$F_{\mu\nu}F^{\mu\nu}$ 中 E_x^2, E_y^2. E_z^2, B^2 都出 2 次.

在 lagange of EM 中,

$$\mathcal{L} = \frac{1}{2}\int \left[(\nabla \times \vec{A})^2 - \left(\frac{\partial \vec{A}}{\partial t}\right)^2 \right] d^3x$$

$$\simeq \frac{1}{2}\int \vec{B}^2 - \vec{E}^2$$

see EM textbook, energy density of $\vec{E}$ field is

$$\boxed{W_e = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3\vec{x}}$$

Thus we have a coefficient ($\frac{1}{2}$).

$$\frac{1}{4}\cdot 2 = \boxed{\frac{1}{2}}$$

thus $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu}$

From Lagrange to EOM

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu}$$

How to obtain the Maxwell eq?

from $\mathcal{L}(q, \dot{q})$

1

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

from $\mathcal{L}(q, \partial_\mu q)$

2

$$\Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} = \frac{\partial \mathcal{L}}{\partial q}$$

from $\mathcal{L}(q_\nu \; \partial_\mu q_\nu)$

3

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_\nu)} = \frac{\partial \mathcal{L}}{\partial q_\nu}}$$

If we have terms $\mathcal{L}(q, \dot{q}, \ddot{q})$ Then it should be more complicated. (see classical mech)

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = ?$$

$$\frac{\partial L}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{4} \frac{\partial (F_{\mu\nu})}{\partial(\partial_\alpha A_\beta)} F^{\mu\nu} \quad \text{--- (1)}$$

$$-\frac{1}{4} F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial(\partial_\alpha A_\beta)} \quad \cdots (2)$$

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial(\partial_\alpha A_\beta)}$$

$$= (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})$$

$$(1) = -\frac{1}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\mu\alpha})F^{\mu\nu}$$

$$= -\frac{1}{4}(F^{\alpha\beta} - F^{\beta\alpha}) = \underbrace{-\frac{1}{2}F^{\alpha\beta}}$$

$$\frac{\partial F^{\mu\nu}}{\partial(\partial_\alpha A_\beta)} \quad \theta$$

using

$$F^{\mu\nu} = \eta^{\mu\mu'}\eta^{\nu\nu'}F_{\mu'\nu'}$$

$$= \left(\eta^{\mu\mu'}\eta^{\nu\nu'} \frac{\partial F_{\mu'\nu'}}{\partial(\partial_\alpha A_\beta)}\right) F_{\mu\nu}$$

$$= \eta^{\mu\mu'}\eta^{\nu\nu'}(\underline{\delta_{\mu'\alpha}}\delta_{\nu'\beta} - \delta_{\mu'\beta}\delta_{\nu'\alpha})F_{\mu\nu}$$

$$= (\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha})F_{\mu\nu}$$

$$= F^{\alpha\beta} - F^{\beta\alpha} = 2F^{\alpha\beta}$$

Thus

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta}}$$

$$-\partial_\alpha F^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(A_\beta)} = -j^{\beta}$$

$$\Leftrightarrow \boxed{\partial_\alpha F^{\alpha\beta} = j^{\beta}}$$

另外2个方程呢?

$$\begin{cases} \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases} \qquad \vec{B} = \nabla\times\vec{A}, \quad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

齐次方程

自动满足

齐次方程.

$$-\nabla\times\frac{\partial}{\partial t}\vec{A} = 0 - \frac{\partial \vec{B}}{\partial t}$$

So 另外2个方程则为恒等式, 在 $(\phi, \vec{A})$ representation 下.

Some History

⓪ $\vec{A}$ may be first introduced by Kelvin

① Maxwell 1861 – 1862 用 $(\phi, \vec{A})$ 20 eqs.

② Heaviside 用 $\vec{E}$, $\vec{B}$ $\Rightarrow$ 20 $\to$ 4 eqs.

③ AB effect 说明 $\vec{A}$ is more fundamental in phys.

用外微分表示：(Hodge的 star / Hodge dual)

$$F = F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$$

$$dF = \partial_\sigma F_{\mu\nu}\, dx^{\sigma} \wedge dx^{\mu} \wedge dx^{\nu}$$

eg

$$\begin{cases} \partial_1 F_{23}\, dx^1 \wedge dx^2 \wedge dx^3 \\ + \partial_2 F_{13}\, dx^2 \wedge dx^1 \wedge dx^3 = \varepsilon^{213}\, dx^1 \wedge dx^2 \wedge dx^3 \\ + \partial_3 F_{12}\, dx^3 \wedge dx^1 \wedge dx^2 = \varepsilon^{312}\, dx^1 \wedge dx^2 \wedge dx^3 \end{cases}$$

$$\Leftrightarrow \quad \underbrace{\varepsilon^{\sigma\mu\nu} \partial_\sigma F_{\mu\nu}}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$\Leftrightarrow \quad \boxed{dF = 0}$$

So 可以用更简单的式表示，下面的另一个表达式 为 $d * F = J$

Hodge star

$$*(dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_p})$$

$$= \frac{1}{(n-p)!}\, \varepsilon_{i_1 \cdots i_p i_{p+1} \cdots i_n}\, dx^{i_{p+1}} \wedge dx^{i_{p+2}} \cdots dx^{i_n}$$

学习理论的方法，先定义 我们计算.

① dx^1, dx^2 $\quad n=2$

$$\begin{cases} *(dx^1) = \frac{1}{(2-1)!}\,\varepsilon_{12}\,dx^2 = dx^2 \\ *(dx^2) = \frac{1}{2!}\,\varepsilon_{21}\,dx^1 = -dx^1 \end{cases}$$

$$\Rightarrow \begin{cases} *dx^1 = dx^2 \\ *dx^2 = -dx^1 \end{cases} \qquad \boxed{*^2 = -1}$$

(2) dx^1, dx^2, dx^3

$$*(dx^1) = \frac{1}{(3-1)!}\left[\varepsilon_{123}\,dx^2\wedge dx^3 + \varepsilon_{132}\,dx^3\wedge dx^2\right]$$

$$= \frac{1}{2}\left(dx^2\wedge dx^3 - dx^3\wedge dx^2\right)$$

$$= dx^2\wedge dx^3$$

$$*(dx^2) = \frac{1}{2!}\left(\varepsilon_{213}\,dx^1\wedge dx^3 + \varepsilon_{231}\,dx^3\wedge dx^1\right)$$

$$= \frac{1}{2!}\left(-dx^1\wedge dx^2 - dx^1\wedge dx^2\right)$$

$$= \underline{-dx^1\wedge dx^3}$$

$$*(dx^2\wedge dx^3) = \frac{1}{(3-2)!}\,\varepsilon_{231}\,dx^1$$

$$= dx^1$$

123
231
312

$$*(dx^1\wedge dx^3) = \frac{1}{}\,\varepsilon_{132}\,dx^3 = -dx^3$$

$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$*F = F_{\mu\nu}\, *(dx^\mu \wedge dx^\nu)$

$= F_{\mu\nu} \frac{1}{2!} \varepsilon^{\mu\nu}{}_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$

$= \frac{1}{2} \tilde{F}_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$

(dual EM tensor

$d(*F) = \partial_\sigma \left(F_{\mu\nu} \frac{1}{2!} \varepsilon^{\mu\nu\alpha\beta} \right) dx^\sigma \wedge dx^\alpha \wedge dx^\beta$

$\left\{ \begin{array}{l} \sigma \neq \alpha, \beta \\ \mu, \nu \in \alpha, \beta. \end{array} \right.$

$\sigma = \mu$ OR $\sigma = \nu$

$= (\partial_\mu F_{\mu\nu}) \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta}\, dx^\mu \wedge dx^\alpha \wedge dx^\beta$

$+ (\partial_\nu F_{\mu\nu}) \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta$

$\boxed{j = j_\alpha\, dx^\alpha}$

$\Leftrightarrow \boxed{d*F = j}$

$\left. \dfrac{*dF \neq d*F}{dF = 0 \quad *0 = 0} \right\} \; *dF \neq d*F$

Magnetic monopole :

$$\begin{cases} \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \end{cases} \Leftrightarrow \boxed{dF = 0}$$

However, $\begin{cases} \nabla\cdot\vec{E} = \rho \\ \nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J} \end{cases} \Leftrightarrow \boxed{d*F = j}$

So we expect for monopole

$$\boxed{d*F = j_e \quad , \quad dF = j_m}$$

Maxwell eq with monopole (set $c=1$, $\mu_0=1$, $\varepsilon_0=1$)

$$\begin{cases} \nabla\cdot\vec{E} = \rho_e \\ \nabla\cdot\vec{B} = \rho_m \\ \nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J}_e \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \vec{J}_m \end{cases}$$

$j_e = (\rho_e \;\; \vec{J}_e)$

$j_m = (\rho_m, \; \vec{J}_m)$

$$\vec{F} = q_e(\vec{E} + \vec{v}\times\vec{B}) + q_m(\vec{B} - \vec{v}\times\vec{E})$$

Please check

$$\begin{cases} \partial_\alpha F^{\alpha\beta} = J_e^\beta \\ \partial_\alpha \tilde{F}^{\alpha\beta} = J_m^\beta \end{cases}$$

Massive EM (proca eq).

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu - \frac{m^2}{2} A^\mu A_\mu$$

$$-\partial_\alpha F^{\alpha\beta} = -j^\beta - \frac{m^2}{2} \frac{\partial}{\partial A_\beta} A^\mu A_\mu$$

$$= -j^\beta - m^2 A^\beta$$

$$\Leftrightarrow \boxed{\partial_\alpha F^{\alpha\beta} = j^\beta + A^\beta}$$

so $\boxed{j \propto A}$

$j^0 \to \rho \quad j^{1-3} \to \vec{J}$

$A^0 \to \phi \quad A^{1-3} \to \vec{A}$

$$\left\{ \begin{array}{l} \boxed{\nabla \cdot \vec{E} = \rho - m^2 \phi} \\ \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J} - m^2 \vec{A} \end{array} \right.$$

如何理解 proca eq：将 $\vec{j} \to (\rho, \vec{J})$

$$\Downarrow$$

$(\rho - m^2\phi, \vec{J} - m^2\vec{A})$ 即可.